THE FIRST NON-VANISHING QUADRATIC TWIST OF AN AUTOMORPHIC L-SERIES

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ABSTRACT. Let f be an automorphic form on $\mathrm{GL}(r)$ for $r = 1, 2$, or 3. Let d be a fundamental discriminant and χ_d the corresponding quadratic Dirichlet character. We consider the question of the least d, relative to the data (level, weight or eigenvalue) of f, such that the central value or derivative of the twisted L-series is nonzero, i.e. $L(1/2, f \otimes \chi_d) \neq 0$ or $L'(1/2, f \otimes \chi_d) \neq 0$.

Let N be the level, say, of f. Using multiple Dirichlet series, we prove the nonvanishing of a central twisted L-value or derivative with $d \ll_\varepsilon N^{1/2+\varepsilon}$ for $\mathrm{GL}(1)$, $d \ll_\varepsilon N^{1+\varepsilon}$ for $\mathrm{GL}(2)$, and $d \ll_\varepsilon N^{2+\varepsilon}$ for $\mathrm{GL}(3)$. We work over $\mathbb{Q}$ for simplicity but the method generalizes to arbitrary number fields.

We conjecture that in all cases there should be such a twist with $d \ll_\varepsilon N^\epsilon$. This would follow from a Lindelof-type bound for a multiple Dirichlet series which does not have an Euler product, but is constructed from a Rankin-Selberg integral applied to automorphic forms which are eigenfunctions of Hecke operators.

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1. Introduction

Let f be an automorphic form on $\mathrm{GL}(r)$ and χ_d a quadratic Dirichlet character of modulus d. A great deal of attention has been paid in recent years to the question of the existence and abundance of fundamental discriminants d such that $L(1/2, f \otimes \chi_d) \neq 0$, that is, such that the L-series of f, twisted by χ_d, does not vanish at the center of the critical strip. In this paper, we ask the following more refined question.

If such a d exists, then what is the least value of $|d|$, relative to the data of f (such as its level N, weight k, or eigenvalue λ), for which the twisted L-series does not vanish at the center? One can sometimes also restrict to fundamental discriminants in certain arithmetic progressions for which such a d does not exist. In this case, it is the non-vanishing at the center of the derivative of the twisted L-function, $L'(1/2, f \otimes \chi_d)$, that is of interest.

1.1. **Level aspect.**

Theorem 1.1. *Let $L(s, f)$ be an automorphic L-series on $GL(r)$ with rank $r = 1$, 2 or 3 and level N. Then*

- *for $r = 1$, there exists some $|d| \ll_\varepsilon N^{1/2+\varepsilon}$,*
- *for $r = 2$, there exists some $|d| \ll_\varepsilon N^{1+\varepsilon}$, and*
- *for $r = 3$, there exists some $|d| \ll_\varepsilon N^{2+\varepsilon}$,*

such that $L(1/2, f \otimes \chi_d) \neq 0$. Restricting to those d for which all central values vanish, the same result holds for $L'(1/2, f \otimes \chi_d) \neq 0$.

It should be noted that for rank $r = 2$ and f holomorphic, by simply combining Waldspurger's theorem [Wal81] with Riemann-Roch, one can show the existence of a $|d| \ll N$ (no epsilon!) such that the central L-value is non-zero. Thus the above result is only interesting in this case for the derivative of the central L-value.

Our proof employs the theory of Multiple Dirichlet Series. Specifically, consider the Dirichlet series in the complex variable w, whose coefficients are themselves (essentially) twisted L-functions of another variable s:

$$Z(s, w) \approx \sum_d \frac{L(s, f \otimes \chi_d)}{d^w}.$$

One can show that the series $Z(s, w)$ has functional equations both as

$$s \mapsto 1 - s$$

and as

$$w \mapsto 1 - w,$$

and moreover that if the rank $r \leq 3$, then the group generated by such reflections is isomorphic to the Weyl group associated to a finite Dynkin diagram. In this way, one can meromorphically continue $Z(s,w)$, originally defined only for $\mathfrak{Re}(s)$ and $\mathfrak{Re}(w)$ sufficiently large, to all of $\mathbb{C}^2$.

If one were to attempt an improvement via these methods on the exponents in the theorem above, a key ingredient would be a "nontrivial" upper bound for $Z(1/2,w)$ at the center of its critical strip, i.e. at $w = 1/2$. The interesting point is that the L-series $Z(1/2,w)$ does **not** have an Euler product. The upper bound that corresponds to the first non-vanishing ranges described above is obtained by a Phragmén-Lindelöf convexity argument. Any improvement on the convexity bound would lead to a corresponding improvement of these results, and a full Lindelöf-type bound would lead to the existence of a non-vanishing twist with

$$|d| \ll_\varepsilon N^\varepsilon.$$

Of course one does not expect a Lindelöf-type bound to be true in general for L-series without an Euler product. See e.g., [CG06] where a counterexample is constructed. However, it does not seem unreasonable to conjecture a Lindelöf type bound for an L-series without an Euler product when that L-series is constructed from a Rankin-Selberg integral applied to one or more automorphic forms that are themselves eigenfunctions of the relevant Hecke operators. Indeed, we conjecture that the double Dirichlet series $Z(s,w)$ satisfies

$$Z(1/2,1/2) \ll_\varepsilon N^\varepsilon.$$

Remark 1.2. For rank $r \geq 4$, the group of functional equations is no longer a finite Weyl group, but is an infinite Coxeter group. Current technology is incapable in this case of obtaining the analytic continuation of $Z(s,w)$ beyond the critical point $(s,w) = (1/2,1)$, the sole exception being the recent work by Bucur and Diaconu [BD08] in the function field analogue.

In particular, one cannot yet answer the following enticing question. Given two automorphic forms f and g on GL(2), each with a positive sign in their functional equation, does there exist a quadratic twist χ_d such that the two twisted L-series simultaneously do not vanish at the center of the critical strip, i.e. $L(1/2, f \otimes \chi_d)L(1/2, g \otimes \chi_d) \neq 0$? Similarly, one cannot yet obtain the second moment of an automorphic form f on GL(2) twisted by quadratic characters, i.e. an asymptotic formula for

$$\sum_{d<X} L(1/2, f \otimes \chi_d)^2, \qquad \text{as } X \to \infty.$$

Remark 1.3. By the Shimura correspondence, the questions raised above for GL(2) are related to questions about twisted moments of half-integral weight forms. Again, the L-series attached to a half-integral weight form $\tilde{f}$ does not have an Euler product, yet it seems likely that if the integral weight Shimura correspondent f is an eigenfunction of the Hecke operators, then $L(s,\tilde{f})$ should satisfy a Lindelöf type bound at the center of its critical strip. In joint work with Gautam Chinta [CHK08], we have observed that, contrary to the integral weight situation, if one forms the multiple Dirichlet series

$$\tilde{Z}(s_1,s_2,w) \approx \sum_d \frac{L(s_1,\tilde{f}\otimes\chi_d)L(s_2,\tilde{f}\otimes\chi_d)}{d^w},$$

then its group of functional equations is isomorphic to the Weyl group associated to the Dynkin diagram A_5, which is finite! Thus we are able to obtain first and second moments for half-integral weight forms twisted by quadratic characters, i.e. asymptotics as $X\to\infty$ for

$$\sum_{d<X} L(1/2,\tilde{f}\otimes\chi_d) \quad \text{and} \quad \sum_{d<X} L(1/2,\tilde{f}\otimes\chi_d)^2.$$

As the first pole of $\tilde{Z}(1/2,1/2,w)$ appears at $w=1$, the second moment is asymptotic to $X\,P(\log X)$, where P is some polynomial. This gives further evidence of the truth of a Lindelöf type bound, even for certain "arithmetic" L-functions without Euler products.

Theorem 1.1 has the following Corollary on simultaneously non-vanishing twists:

Corollary 1.4. *Let $L(s,\chi_{N_1}), L(s,\chi_{N_2}), L(s,\chi_{N_3})$ be three Dirichlet L-series with conductors N_1,N_2,N_3. Let $L(s,f)$ be an irreducible L-series on $GL(2)$ with analytic conductor N. Then*

(1) *there exists $|d|\ll(N_1N_2)^{1+\epsilon}$ with*

$$L(1/2,\chi_d\chi_{N_1})L(1/2,\chi_d\chi_{N_2})\neq 0,$$

(2) *there exists $|d|\ll(N_1N_2N_3)^{2+\epsilon}$ such that*

$$L(1/2,\chi_d\chi_{N_1})L(1/2,\chi_d\chi_{N_2})L(1/2,\chi_d\chi_{N_3})\neq 0,$$

(3) *there exists $|d|\ll(N_1N)^{2+\epsilon}$ such that*

$$L(1/2,\chi_d\chi_{N_1})L(1/2,f\otimes\chi_d)\neq 0.$$

1.2. **Weight and Eigenvalue aspects.** The weight and eigenvalue aspects of this question are dealt with in a similar fashion. The difficulties in the above analysis coming from handling non-archimedian places are now relegated to the archimedian place, where one simply tracks Gamma factors and applies Stirling's formula.

For rank $r = 2$, one can again apply Waldspurger's theorem and arguments dating back to Maass.[1] These imply that an automorphic form of even weight k has a nonvanishing central twist χ_d with $|d| \ll k$, and that a Maass form of eigenvalue $\lambda = \frac{1}{4} + t^2$ has a nonvanishing twist with $|d| \ll t \approx \lambda^{1/2}$. Our methods match this for the twisted derivative:

Theorem 1.5. *Let f be a* GL(2) *holomorphic form of weight k, or a Maass form of eigenvalue λ. Then restricting to an arithmetic progression of fundamental discriminants for which the twisted central L-value vanishes, there exists some d in this progression with*

$$|d| \ll_\varepsilon k^{1+\varepsilon}, \quad \text{or} \quad |d| \ll_\varepsilon \lambda^{1/2+\varepsilon},$$

such that the twisted central derivative $L'(1/2, f \otimes \chi_d) \neq 0$.

Remark 1.6. It is interesting to note that while the "conductor" in the level aspect is N, in the weight aspect it is k^2. Despite the fact that "convexity" arguments apply identically to both aspects, it is the dimension of the space (being $\sim N$ or $\sim k$), not the conductor, which plays the role of being the convex bound.

For a tempered Maass form on GL(3) of type (ν_1, ν_2), where $\nu_j = 1/3 + it_j$, and $t_j \in \mathbb{R}$, $j = 1, 2$, the eigenvalue of the Laplace-Beltrami operator is $\lambda = 1 + 3t_1^2 + 3t_1t_2 + 3t_2^2$. Our methods show

Theorem 1.7. *Let f be a Maass form on* GL(3) *of eigenvalue λ. Then there exists some d with*

$$|d| \ll_\varepsilon \lambda^{3/2+\varepsilon},$$

such that the twisted central value $L(1/2, f \otimes \chi_d)$ is non-zero.

Remark 1.8. The proofs of Theorems 1.5 and 1.7 follow in a straightforward way from the general framework which proves the level aspect analogue, Theorem 1.1. The changes are sufficiently minor that we omit the details. Similarly, statements about central L-values are easily altered to handle central derivatives; these details will also be suppressed.

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[1] P. Sarnak pointed us to the thesis of J. Huntley where these ideas are vastly generalized.

1.3. **The "Moment Approach".** If one wishes to avoid analysis in several complex variables, one can of course mimic the Multiple Dirichlet Series approach, using the approximate functional equation, Gauss sums, and Poisson summation in place of multi-variate meromorphic continuation. One then finds an exact expression relating a main term to a linear combination of sums of central twisted L-values, and from this extracts the same non-vanishing results.

Alternatively, one could consider simply computing a first moment of the central twisted L-value, that is, obtaining a main term and an error term; see e.g. [Iwa90, Sou00]. Using this "Moment Approach," one requires a certain careful Siegel zero analysis, and Heath-Brown's real character sum estimate [HB95]. Because Heath-Brown's estimate is only known over $\mathbb{Q}$, the above approach does not currently generalize to arbitrary number fields.

As far as we can tell, this yields results which either match those from Multiple Dirichlet Series (MDS) or are slightly worse. The following table catalogs our findings. Suppressing ε factors, the specified method proves a nonvanishing twist or twisted derivative with $|d|$ not exceeding the given power of N, k or λ.

Rank r	Aspect	Moment Approach	MDS
$r=1$	Level N	$N^{1/2}$	$N^{1/2}$
$r=2$	Level N	N^{1}	N^{1}
$r=3$	Level N	$N^{19/5}$	N^{2}
$r=2$	Weight k	$k^{3/2}$	k^{1}
$r=2$	Eigenvalue λ	$\lambda^{3/4}$	$\lambda^{1/2}$
$r=3$	Eigenvalue λ	$\lambda^{21/10}$	$\lambda^{3/2}$

Both Multiple Dirichlet Series and classical methods have so far failed to provide much insight on the rank $r \geq 4$ situation, cf. Remark 1.2.

1.4. **Outline.** This paper is organized as follows. In §2, we provide an overview of the main argument in the level aspect. In §3, we detail the interchange property required for iterating various functional equations. Then in §4, we detail the proof of the functional equations required for Theorem 1.1 in the simplest case of rank $r=1$ when the level N is prime. In §5, we sketch the argument in the general case, before completing the proof of Theorem 1.1 in §6. We conclude in §7 with a sample analysis of the "Moment Approach".

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2. A SKETCH OF THE ARGUMENT

In this section we present an overview of the argument in the level aspect. It will contain some very imprecise statements regarding functional equations but should nevertheless be a useful reference guide when going through the actual proof in the latter sections. We pretend throughout this section, for clarity of exposition, that all numbers are positive and congruent to 1 modulo 4, and that quadratic reciprocity is perfect.

2.1. Determining the Functional Equations.

Recall the notation from §1. The L-series associated to f is

$$L(s,f)=\sum_{n\geq 1}^{\infty}\frac{c(n)}{n^s},$$

with the Euler product

$$L(s,f)=\prod_p\prod_{i=1}^{r}\left(1-\alpha_p^{(i)}p^{-s}\right)^{-1},$$

and functional equation

$$\Lambda(s,f):=N^{s/2}G_f(s)L(s,f)=\epsilon_f\Lambda(1-s,\tilde{f}) \tag{2.1}$$

Here $|\epsilon_f|=1$ is the root number, N is the level,

$$G_f(s)=\pi^{-rs/2}\prod_{j=1}^{r}\Gamma\left(\frac{s+\kappa_j}{2}\right)$$

is a product of archimedean gamma factors, and $\tilde{f}$ is the contragradient of f. The twisted L-series has Euler product

$$L(s,f\otimes\chi_d)=\sum_n\frac{c(n)\chi_d(n)}{n^s}=\prod_p\prod_{i=1}^{r}\left(1-\chi_d(p)\alpha_p^{(i)}p^{-s}\right)^{-1}, \tag{2.2}$$

and if $(N,d)=1$ then its functional equation is given by

$$\Lambda(s,f\otimes\chi_d):=(N|D|^r)^{s/2}G_{d,f}(s)L(s,f\otimes\chi_d)=\epsilon_f\psi(d)\Lambda(s,\tilde{f}\otimes\chi_d) \tag{2.3}$$

Here $D=4d$ or $D=d$ is the conductor of χ_d, $G_{d,f}$ is a product of gamma factors depending only on f and the sign of d, and ψ is some character modulo N.

Consider the following double Dirichlet series:

$$Z_1(s,w)=\sum_d\frac{L(s,f\otimes\chi_d)}{d^w}. \tag{2.4}$$

When d is square free, $L(s, f \otimes \chi_d)$ is the twisted L-series of (2.2). When d is not square free, $L(s, f \otimes \chi_d)$ is the modified L series given by (3.3) (taking $M = a_1 = l_1 = 1$ in this simplified setting). Very roughly, inserting (2.2) into (2.4), $Z_1(s, w)$ is represented by the double Dirichlet series

$$\sum_{d,n} \frac{c(n)\chi_d(n)}{d^w n^s}.$$

This suggests that if quadratic reciprocity held perfectly, that is $\chi_d(n) = \chi_n(d)$, then we could rewrite this as

$$Z_1(s, w) = \sum_n \frac{L(w, \chi_n)c(n)}{n^s}, \tag{2.5}$$

and in fact this interchange is allowed by equation (3.5).

Dualizing the numerator of (2.5) and suppressing gamma factors, we see that there is a functional equation sending

$$Z_1(s, w) \to Z_1(s + w - 1/2, 1 - w). \tag{2.6}$$

On the other hand, if we apply (2.3) to the numerator of (2.4), we find that there is a functional equation sending

$$Z_1(s, w) \to N^{1/2-s} Z_2(1 - s, w + rs - r/2), \tag{2.7}$$

where

$$Z_2(s, w) = \sum \frac{L(s, \tilde{f} \otimes \chi_d)\psi(d)}{d^w} = \sum_{n,d} \frac{\chi_n(d)\psi(d)\tilde{c}(n)}{n^s d^w}. \tag{2.8}$$

Applying the interchange property (3.5) to (2.8) we find that

$$Z_2(s, w) = \sum_n \frac{L(w, \chi_n \psi)\tilde{c}(n)}{n^s}.$$

Dualizing this numerator we see that there is a functional equation

$$Z_2(s, w) \to N^{1/2-w} Z_2(s + w - 1/2, 1 - w). \tag{2.9}$$

Similarly, (2.7) can be used in reverse to give

$$Z_2(s, w) \to N^{1/2-s} Z_1(1 - s, w + rs - r/2). \tag{2.10}$$

2.2. **Iterating Functional Equations.** We now apply these functional equations in sequence. If the rank $r = 1$, we apply in succession (2.6), (2.7), and (2.9), obtaining

$$\begin{aligned} Z_1(s, w) &\to Z_1(s + w - 1/2, 1 - w) \to N^{1-s-w} Z_2(3/2 - s - w, s) \\ &\to N^{3/2-2s-w} Z_2(1 - w, 1 - s). \end{aligned} \tag{2.11}$$

Remark 2.12. On GL(1) there is an extra symmetry in (2.8), namely $Z_2(s,w) \approx Z_2(w,s)$, coming from the relation $\psi(d) \approx c(d)$.

If $r = 2$ we apply in succession (2.6), (2.7), (2.9) and (2.10), obtaining

$$\begin{aligned} Z_1(s,w) &\to Z_1(s+w-1/2, 1-w) \to N^{1-s-w} Z_2(3/2-s-w, w+2s-1) \\ &\to N^{5/2-3s-2w} Z_2(s, 2-2s-w) \\ &\to N^{3-4s-2w} Z_1(1-s, 1-w). \end{aligned}$$

If $r = 3$ we apply in succession (2.6), (2.7), (2.9), (2.10), (2.6) and (2.7), obtaining

$$\begin{aligned} Z_1(s,w) &\to Z_1(s+w-1/2, 1-w) \\ &\to N^{1-s-w} Z_2(3/2-s-w, 3s+2w-2) \\ &\to N^{7/2-4s-3w} Z_2(2s+w-1, 3-3s-2w) \\ &\to N^{5-6s-4w} Z_1(2-2s-w, w+3s-3/2) \\ &\to N^{5-6s-4w} Z_1(s, 5/2-3s-w) \\ &\to N^{11/2-7s-4w} Z_2(1-s, 1-w). \end{aligned}$$

Remark 2.13. In each of the cases above, enough iterations of the functional equations will return us to $Z_1(s,w)$. For rank $r \geq 4$, one can cycle the dualizations ad infinitum, never arriving at the desired argument $Z(1-s, 1-w)$.

It is explained in complete detail in [DGH03] how the meromorphic continuation of $Z_1(s,w)$ and $Z_2(s,w)$ is obtained from the above functional equations, together with the known polar behavior of the numerators. Restricting to $s = 1/2$, in every case the series $Z_1(1/2, w)$ will have a pole of order at least 1 at $w = 1$.

When the functional equations above are applied to $Z_1(1/2, w)$ we find the following relations hold: When $r = 1$:

$$Z_1(1/2, w) \to N^{1/2-w} Z_2(1/2, 1-w),$$

when $r = 2$:

$$Z_1(1/2, w) \to N^{1-2w} Z_1(1/2, 1-w),$$

and when $r = 3$:

$$Z_1(1/2, w) \to N^{2-4w} Z_2(1/2, 1-w).$$

The existence of a finite order pole at $w = 1$ is the basis of the non-vanishing argument presented in §6. This argument will show that it is precisely the three exponents above which lead to nonvanishing twists with $|d| \ll N^{1/2+\varepsilon}$ for $r = 1$, $|d| \ll N^{1+\varepsilon}$ for $r = 2$, and $|d| \ll N^{2+\varepsilon}$ for $r = 3$.

Roughly speaking, the squares of these exponents are essentially the "conductors" of the L-series $Z_1(1/2, w)$, and one can express an L-series using a finite Dirichlet polynomial whose length is square root of the conductor. Then since $Z_1(1/2, w)$ has a pole at $w = 1$, one easily derives a contradiction if all of the coefficients of this Dirichlet polynomial vanish.

3. The Interchange Property

When all primes are included in the product (2.2) the functional equation (2.3) has its optimal form. However, it is often convenient to omit factors corresponding to "bad" primes, for example those contained in S, a finite set of primes including 2 and those primes dividing N.

Let $M = \prod_{p \in S} p$. For such M, S we denote the L-series with Euler factors corresponding to primes dividing M removed as follows:

$$L_M(s,f) = \prod_{p \notin S} \prod_{i=1}^{r} \left(1 - \alpha_p^{(i)} p^{-s}\right)^{-1} = L(s,f) \prod_{p \in S} \prod_{i=1}^{r} \left(1 - \alpha_p^{(i)} p^{-s}\right). \quad (3.1)$$

When twisted by χ_d, the L-series $L(s, f \otimes \chi_d)$ will have a perfect functional equation of the form (2.3) when χ_d is a primitive character. This corresponds to the case where d is square free. When d is *not* square free, it is possible to complete $L(s, f \otimes \chi_d)$ by multiplying by a certain Dirichlet polynomial in such a way that the resulting product has a functional equation of precisely the same form (2.3). (For the simplest example, see [GH85], [CFH06].) In fact some very stringent additional conditions can be imposed on the Dirichlet polynomial.

To be more precise, let $l_1, l_2 > 0, l_1, l_2 | M$ and $a_1, a_2 \in \{1, -1\}$ and let $\chi_{a_1 l_1}, \chi_{a_2 l_2}$ be the quadratic characters corresponding to $a_1 l_1, a_2 l_2$. We formulate the following collection of properties for two classes of Dirichlet polynomials associated to f.

Property 3.2. *For n, d positive integers, $(nd, M) = 1$, write $d = d_0 d_1^2, n = n_0 n_1^2$, with d_0, n_0 square free. Let $c(n)$ denote the coefficients of $L(s,f)$. Let $P_{d_0,d_1}^{(a_1 l_1)}(s), Q_{n_0,n_1}^{(a_2 l_2)}(w)$ be Dirichlet polynomials defined by*

$$P_{d_0,d_1}^{(a_1 l_1)}(s) = \prod_{p^\alpha || d_1} \left(1 + a_{d_0,p}^{(\alpha)} p^{-s} + \cdots + a_{d_0,p^{2r\alpha}}^{(\alpha)} p^{-2r\alpha s}\right)$$

and

$$c(n_0 n_1^2) Q_{n_0,n_1}^{(a_2 l_2)}(w) = c(n_0 n_1^2) \prod_{p^\beta || n_1} \left(1 + b_{n_0,p}^{(\beta)} p^{-w} + \cdots + b_{n_0,p^{2\beta}}^{(\beta)} p^{-2\beta w}\right).$$

We say that P, Q satisfy the conditions of the Property 3.2 if the two functional equations

$$d_1^{rs} P_{d_0,d_1}^{(a_1 l_1)}(s) = d_1^{r(1-s)} P_{d_0,d_1}^{(a_1 l_1)}(1-s),$$

and

$$n_1^w c(n_0 n_1^2) Q_{n_0,n_1}^{(a_2 l_2)}(w) = n_1^{1-w} c(n_0 n_1^2) Q_{n_0,n_1}^{(a_2 l_2)}(1-w)$$

hold, and if in addition the following interchange of summation is valid for s and w having sufficiently large real part:

$$\sum_{(d,M)=1} \frac{L_M(s, f \otimes \chi_{d_0} \chi_{a_1 l_1}) \chi_{a_2 l_2}(d_0) P_{d_0,d_1}^{(a_1 l_1)}(s)}{d^w}$$
$$= \sum_{(n,M)=1} \frac{L_M(w, \chi_{n_0} \chi_{a_2 l_2}) \chi_{a_1 l_1}(n_0) c(n_0 n_1^2) Q_{n_0,n_1}^{(a_2 l_2)}(w)}{n^s}.$$

Here χ_{n_0} denotes the quadratic character with conductor n_0 defined by $\chi_{n_0}(*) = \left(\frac{*}{n_0}\right)$. (Recall $2|M$ so $(2, n_0) = 1$.)

There is a rather large amount of notation involved in the interchange of summation formula above. To make this a little easier on the eyes, we define

$$L_M(s, f \otimes \chi_d \chi_{a_1 l_1}) = L_M(s, f \otimes \chi_{d_0} \chi_{a_1 l_1}) P_{d_0,d_1}^{(a_1 l_1)}(s) \tag{3.3}$$

and

$$L_M(w, \chi_n \chi_{a_2 l_2}) = L_M(w, \chi_{n_0} \chi_{a_2 l_2}) Q_{n_0,n_1}^{(a_2 l_2)}(w). \tag{3.4}$$

Thus the interchange in the order of summation above takes the slightly more reasonable form:

$$\sum_{(d,M)=1} \frac{L_M(s, f \otimes \chi_d \chi_{a_1 l_1}) \chi_{a_2 l_2}(d)}{d^w} = \sum_{(n,M)=1} \frac{L_M(w, \chi_n \chi_{a_2 l_2}) \chi_{a_1 l_1}(n) c(n)}{n^s}. \tag{3.5}$$

4. The simplest case, $r = 1$ and level N prime

In this section we follow through in more detail the reflections of (2.11) in the simplest case that $N = p$, a prime. As f is on $\mathrm{GL}(1)/\mathbb{Q}$, it is simply some Dirichlet character $\theta (\operatorname{mod} p)$, the coefficients $c(n) = \theta(n)$, and the contragredient $\tilde{f}$ has coefficients $\tilde{c}(n) = \bar{\theta}(n)$. We keep the general notation for uniformity with the higher rank case. Although in this case we should set $M = 2p$, we will ignore the harmless role of the prime 2. We begin with the double Dirichlet series

$$Z(s,w;f) = \sum_{(n,p)=1} \frac{L(w,\chi_{n_0})c(n_0 n_1^2) Q^{(1)}_{n_0,n_1}(w)}{n^s},$$

where recall $n = n_0 n_1^2$ and n_0 is square-free. This series converges absolutely for any fixed value of $w \neq 1$ as long as the real part of s is sufficiently large. Following the notation of (3.4) we rewrite this more compactly as

$$\begin{aligned} Z(s,w;f) &= \sum_{(n,p)=1} \frac{L(w,\chi_n)c(n)}{n^s} = \sum_{(n,p)=1} \frac{L_p(w,\chi_n)(1-\chi_n(p)p^{-w})^{-1}c(n)}{n^s} \\ &= (1-p^{-2w})^{-1} \sum_{(n,p)=1} \frac{L_p(w,\chi_n)(1+\chi_n(p)p^{-w})c(n)}{n^s}, \end{aligned}$$

where we put in and took out the p-part of L, cf. (3.1).

Simplifying gives

$$(1-p^{-2w})Z(s,w;f) = \sum_{(n,p)=1} \frac{L_p(w,\chi_n)c(n)}{n^s} + \frac{1}{p^w} \sum_{(n,p)=1} \frac{L_p(w,\chi_n)\chi_n(p)c(n)}{n^s}. \tag{4.1}$$

Notice that applying the interchange property to (4.1) gives

$$(1-p^{-2w})Z(s,w;f) = \sum_{(d,p)=1} \frac{L_p(s, f\otimes\chi_d)}{d^w} + \sum_{(d,p)=1} \frac{L_p(s, f\otimes\chi_d\chi_p)}{(dp)^w}. \tag{4.2}$$

Remark 4.3. It is crucial that we start not with just $Z(s,w;f)$ but with the factor $(1-p^{-2w})$ in front! Otherwise certain coefficients $\alpha_j(w)$ in (6.2) will not be $\ll N^\varepsilon$, but will instead be bounded by about N^2. This would drastically worsen the first non-vanishing results. See also Remark 5.5.

Returning to (4.1), we must reinsert the p-parts before applying the functional equation of $L(w,\chi_n)$.

$$\begin{aligned} (1-p^{-2w})Z(s,w;f) &= \sum_{(n,p)=1} \frac{L(w,\chi_n)(1-\chi_n(p)p^{-w})c(n)}{n^s} \\ &\quad + \frac{1}{p^w} \sum_{(n,p)=1} \frac{L(w,\chi_n)(1-\chi_n(p)p^{-w})\chi_n(p)c(n)}{n^s}. \end{aligned}$$

Now we apply the functional equation of $L(w, \chi_n)$ and suppress Gamma factors:

$$\begin{aligned}
& (1-p^{-2w})Z(s,w;f) \\
\to \quad & \sum_{(n,p)=1} \frac{n^{1/2-w}L(1-w,\chi_n)(1-\chi_n(p)p^{-w})c(n)}{n^s} \\
& + \frac{1}{p^w} \sum_{(n,p)=1} \frac{n^{1/2-w}L(1-w,\chi_n)(1-\chi_n(p)p^{-w})\chi_n(p)c(n)}{n^s} \\
= \quad & \sum_{(n,p)=1} \frac{L_p(1-w,\chi_n)(1-\frac{\chi_n(p)}{p^{1-w}})^{-1}(1-\chi_n(p)p^{-w})c(n)}{n^{s+w-1/2}} \\
& + \sum_{(n,p)=1} \frac{L_p(1-w,\chi_n)(1-\frac{\chi_n(p)}{p^{1-w}})^{-1}(1-\chi_n(p)p^{-w})\chi_n(p)c(n)}{p^w n^{s+w-1/2}} \\
= \quad & (1-\frac{1}{p^{2-2w}})^{-1} \left(\sum_{(n,p)=1} \frac{L_p(1-w,\chi_n)(1+\frac{\chi_n(p)}{p^{1-w}})(1-\chi_n(p)p^{-w})c(n)}{n^{s+w-1/2}} \right. \\
& \left. + \sum_{(n,p)=1} \frac{L_p(1-w,\chi_n)(1+\frac{\chi_n(p)}{p^{1-w}})(1-\chi_n(p)p^{-w})\chi_n(p)c(n)}{p^w n^{s+w-1/2}} \right).
\end{aligned}$$

Collecting terms gives

$$\begin{aligned}
= \quad & (1-p^{-2+2w})^{-1}\left(1-p^{-2w}\right) \\
& \times \left(\sum_{(n,p)=1} \frac{L_p(1-w,\chi_n)c(n)}{n^{s+w-1/2}} + \sum_{(n,p)=1} \frac{L_p(1-w,\chi_n)c(n)\chi_n(p)}{n^{s+w-1/2}p^{1-w}} \right) \\
= \quad & (1-p^{-2+2w})^{-1}\left(1-p^{-2w}\right) \\
& \times \left(\sum_{(d,p)=1} \frac{L(s+w-1/2, f\otimes\chi_d)}{d^{1-w}} + \sum_{(d,p)=1} \frac{L(s+w-1/2, f\otimes\chi_d\chi_p)}{(dp)^{1-w}} \right),
\end{aligned}$$

upon applying the interchange property (and using the fact that the p parts of f and χ_p are trivial).

Dualize the L-functions in the numerator:

$$\begin{aligned}
\to\quad & (1-p^{-2+2w})^{-1}\left(1-p^{-2w}\right)\Bigg(\sum_{(d,p)=1}\frac{N^{1-s-w}d^{1-s-w}L(3/2-s-w,\tilde f\otimes\chi_d)}{d^{1-w}} \\
& + \sum_{(d,p)=1}\frac{N^{1-s-w}d^{1-s-w}L(3/2-s-w,\tilde f\otimes\chi_d\chi_p)}{(dp)^{1-w}}\Bigg) \\
= \quad & (1-p^{-2+2w})^{-1}\left(1-p^{-2w}\right)N^{1-s-w} \\
& \times\left(\sum_{(d,p)=1}\frac{L(3/2-s-w,\tilde f\otimes\chi_d)}{d^s}+\sum_{(d,p)=1}\frac{L(3/2-s-w,\tilde f\otimes\chi_d\chi_p)}{(dp)^s}\right).
\end{aligned}$$

Returning to (4.2), combine the above computations, and specialize to $s=1/2$. Collect the appropriate Gamma factors, and immediately hide them and the factors $(1-p^{-2w})$, etc., in some coefficients α_j, $\tilde\alpha_{\tilde j}$:

$$\begin{aligned}
& N^{w/2}\left(\alpha_1(w)\sum_{(d,p)=1}\frac{L_p(1/2,f\otimes\chi_d)}{d^w}+\alpha_2(w)\sum_{(d,p)=1}\frac{L_p(1/2,f\otimes\chi_d\chi_p)}{(dp)^w}\right) \\
= \quad & N^{(1-w)/2}\Bigg(\tilde\alpha_1(1-w)\sum_{(d,p)=1}\frac{L(1-w,\tilde f\otimes\chi_d)}{d^s} \\
& + \tilde\alpha_2(1-w)\sum_{(d,p)=1}\frac{L(1-w,\tilde f\otimes\chi_d\chi_p)}{(dp)^s}\Bigg). \qquad (4.4)
\end{aligned}$$

This is the information assembled for the more general setting in equation (6.2).

5. Good primes, bad primes, and functional equations

We now resketch the calculations above in the more general setting. The main tools we will use are the analytic continuation in (s,w), and an estimate for the growth in vertical strips $w=\sigma+it$, for fixed σ and s, of the double Dirichlet series

$$Z^{(l,l')}(s,w;f)=\sum_{(n,M)=1}\frac{L(w,\tilde\chi_{n_0}\chi_{l'})\chi_l(n_0)c(n_0n_1^2)Q^{(l')}_{n_0,n_1}(w)}{n^s}. \qquad (5.1)$$

Following the notation of (3.4) we rewrite this more compactly as

$$Z^{(l,l')}(s,w;f)=\sum_{(n,M)=1}\frac{L(w,\tilde{\chi}_n\chi_{l'})\chi_l(n)c(n)}{n^s}. \tag{5.2}$$

This series converges absolutely for any fixed value of $w\neq 1$ as long as the real part of s is sufficiently large. Here l,l' can take the values $1,-1,2,-2$ so we are actually finding the analytic continuation of 16 separate series. (Recall $\tilde{\chi}_{n_0}$ denotes the quadratic character with conductor n_0 defined by $\tilde{\chi}_{n_0}(*)=\left(\frac{*}{n_0}\right)$.) We sum over $n>0$ and use the decomposition $n=n_0n_1^2$, with n_0 square free. A key point is that although we sum over n relatively prime to M, the L-series in the numerator do *not* have the factors corresponding to primes dividing M removed. That is, they are complete L-series and satisfy the functional equations (2.3). Because of this, $Z^{(l,l')}$ does *not* satisfy Property 3.2.

The analytic properties of $Z^{(l,l')}(s,w;f)$ are found by studying the building blocks from which it is constructed: For $l_1,l_2>0, l_1,l_2|M$ and $a_1,a_2\in\{1,-1\}$ we define (referring to (3.3))

$$Z_M(s,w;\chi_{a_2l_2},\chi_{a_1l_1};f)=\sum_{(d,M)=1}\frac{L_M(s,f\otimes\chi_d\chi_{a_1l_1})\chi_{a_2l_2}(d)}{d^w}. \tag{5.3}$$

For fixed s ($s\neq 1$) this converges absolutely if the real part of w is sufficiently large. Because the L-series in the numerator has the primes dividing M removed, and because the sum is over d relatively prime to M, the series $Z_M(s,w;\chi_{a_2l_2},\chi_{a_1l_1};f)$ satisfies the interchange Property 3.2. Thus we have the alternate expression:

$$Z_M(s,w;\chi_{a_2l_2},\chi_{a_1l_1};f)=\sum_{(n,M)=1}\frac{L_M(w,\tilde{\chi}_n\chi_{a_2l_2})\chi_{a_1l_1}(n)c(n)}{n^s}. \tag{5.4}$$

Because of these two expressions $Z_M(s,w;\chi_{a_2l_2},\chi_{a_1l_1};f)$ has two functional equations: one from (5.3) in s that transforms it into a linear combination of

$$Z_M(1-s,w+rs-1;\chi_{a_2l_2},\chi_{a_1l_1};f);$$

and the other, from (5.4) is in w, transforming it into a linear combination of

$$Z_M(s+w-1/2,1-w;\chi_{a_2l_2},\chi_{a_1l_1};f),$$

both over varying a_1,l_1,a_2,l_2. The abstract analytic continuation of these building blocks has been written down in detail for the cases $r=1,2,3$ in several places. See, for example, [BFH96], [DGH03], [BBC$^+$06]. What we do here, which is slightly different, is sketch the analytic continuation while keeping careful track of the primes dividing

M in the growth estimates. Thus we will keep details to a minimum but make sure that the bounds we obtain have an explicit dependence on M.

Remark 5.5. A similar computation appears in [CD05], see their Proposition 3.4, for $r = 3$ and f self-dual. Their starting point is not the imperfect series of (5.2), and as a result, their bounds are weaker (compare their bound M^5 in their equation (3.1) against our bound of M^3 for this case in Proposition 5.7 below). This is precisely the reason for our imperfect starting point of (5.2). See also Remark 4.3.

The series we are really interested in is $Z^{(l,l')}(s,w;f)$, defined in (5.2), with the order of summation interchanged, that is, with quadratic twists of automorphic L-series in the numerator. This interchange, however, can not be done easily because of the presence of Euler factors corresponding to primes dividing M. To overcome this difficulty we first write

$$Z^{(l,l')}(s,w;f) = \sum_{(n,M)=1} \frac{L_M(w,\tilde{\chi}_n\chi_{l'})\prod_{p|M}\left(1-\tilde{\chi}_{n_0}(p)\chi_{l'}(p)p^{-w}\right)^{-1}\chi_l(n)c(n)}{n^s}.$$

Thus

$$Z^{(l,l')}(s,w;f)\prod_{p|M}\left(1-p^{-2w}\right) = \sum_{(n,M)=1} \frac{L_M(w,\tilde{\chi}_n\chi_{l'})\prod_{p|M}\left(1+\tilde{\chi}_{n_0}(p)\chi_{l'}(p)p^{-w}\right)\chi_l(n)c(n)}{n^s}.$$

The right hand side is now a linear combination of $Z_M(s,w;\chi_{l'},\chi_l;f)$:

$$Z^{(l,l')}(s,w;f)\prod_{p|M}\left(1-p^{-2w}\right) = \sum_{m|M}\chi_{l'}(m)m^{-w}Z_M(s,w;\chi_{l'},\chi_l\chi_m;f).$$

Interchanging the order of summation using Property 3.2, we see that each $Z_M(s,w;\chi_{l'},\chi_l\chi_m;f)$ can be written as follows:

$$Z_M(s,w;\chi_{l'},\chi_l\chi_m;f) = \sum_{(d,M)=1}\frac{L_M(s,f\otimes\chi_d\chi_l\chi_m)\chi_{l'}(d)}{d^w}.$$

Combining the above equations we have the fundamental relation

$$\sum_{m|M}\chi_{l'}(m)\sum_{(d,M)=1}\frac{L_M(s,f\otimes\chi_d\chi_l\chi_m)\chi_{l'}(d)}{(dm)^w} \tag{5.6}$$
$$= Z^{(l,l')}(s,w;f)\prod_{p|M}\left(1-p^{-2w}\right).$$

The left hand side of (5.6) is what is required for our application. Analytic bounds for it will follow from analytic bounds for $Z^{(l,l')}(s,w;f)$.

Our goal is to prove

Proposition 5.7. *The function $Z^{(l,l')}(s,w;f)$ defined in* (5.2) *has a meromorphic continuation to all $s,w\in\mathbb{C}^2$. Furthermore, set $s=1/2$; then $Z^{(l,l')}(1/2,w;f)$ is analytic except for a pole of finite order at $w=1$, and possible poles at $w=3/4,1/4$ when $r=3$.*

Proof. We begin by applying the functional equation in w to the numerator of $Z^{(l,l')}(s,w;f)$. This is (2.3) with $r=1$ and s replaced by w.

This gives us, after pulling out the Euler factors dividing M

$$Z^{(l,l')}(s,w;f) = \sum_{a=1,-1}\sum_{(n,M)=1}\frac{G_a(1-w)}{G_a(w)}L_M(1-w,\tilde{\chi}_n\chi_{l'})$$
$$\times\frac{\chi_l(n)c(n)\prod_{p|M}(1-\tilde{\chi}_n(p)\chi_{l'}(p)p^{-1+w})^{-1}}{n^{s+w-1/2}}.$$

Here the fraction $\frac{G_a(1-w)}{G_a(w)}$ is a ratio of gamma factors that depends only on the congruence class of n modulo 4. Multiplying both sides by $\prod_{p|M}(1-p^{-2+2w})$ yields

$$Z^{(l,l')}(s,w;f)\prod_{p|M}(1-p^{-2+2w}) = \sum_{a=1,-1}\sum_{(n,M)=1}\frac{G_a(1-w)}{G_a(w)}L(1-w,\tilde{\chi}_n\chi_{l'})$$
$$\times\frac{\chi_l(n)c(n)\prod_{p|M}(1+\tilde{\chi}_n(p)\chi_{l'}(p)p^{-1+w})}{n^{s+w-1/2}}.$$

As explained in [DGH03], using χ_l to sieve the n modulo 4 enables us to isolate and recombine sums over n in congruence classes 1 and -1 modulo 4 to produce a linear combination of $Z_M(s,w;\chi_{l'},\chi_l\chi_m;f)$ with different l on the right hand side above. We now use Stirling's formula to estimate the ratios of Gamma factors. Specifically, for $\sigma_1>\sigma_2$ and t real and $|t|$ large,

$$|G_a(\sigma_1+it)/G_a(\sigma_2-it)|\ll(|t|+1)^{(\sigma_1-\sigma_2)/2}, \tag{5.8}$$

with the implied constant independent of d and N.

The above relation can be more compactly represented by

$$\begin{aligned} & Z^{(l,l')}(s,w;f)\prod_{p|M}(1-p^{-2+2w}) \qquad (5.9) \\ &= \sum_{\substack{m|M \\ (m,2)=1}} m^{-1+w} \sum_{l''} A^{(1)}_{m,l,l''}(w) Z_M(s+w-1/2, 1-w; \chi_{l'}, \chi_{l''}\chi_m; f), \end{aligned}$$

where the coefficients $A^{(1)}_{m,l,l''}(w)$ are analytic and satisfy the upper bound

$$A^{(1)}_{m,l,l''}(-\epsilon - it) \ll (1+|t|)^{1+3\epsilon}.$$

We will now obtain similar upper bounds for $Z_M(s,w;\chi_{l'},\chi_l\chi_m;f)$ after reflecting the s variable to $1-s$.

We begin by writing

$$\begin{aligned} Z_M(s,w;\chi_{l'},\chi_l\chi_m;f) &= \sum_{(d,M)=1} \frac{L(s, f\otimes \chi_d\chi_l\chi_m)\chi_{l'}(d)}{d^w} \\ & \quad \times \prod_{p|M}\prod_{i=1}^{r}\left(1-\chi_d(p)\chi_l(p)\chi_m(p)\alpha_p^{(i)}p^{-s}\right). \end{aligned}$$

that is, we put back in and take out the factors dividing M. We now apply the functional equation in s, pull out the Euler factors corresponding to primes dividing M, and multiply both sides by

$$\prod_{p|(M/m)}\prod_{i=1}^{r}\left(1-(\tilde{\alpha}_p^{(i)})^2p^{-2+2s}\right)$$

to clear denominators. (Note that primes dividing m do not appear in this product.) We obtain

$$\begin{aligned} & Z_M(s,w;\chi_{l'},\chi_l\chi_m;f)\prod_{p|(M/m)}\prod_{i=1}^{r}\left(1-(\tilde{\alpha}_p^{(i)})^2p^{-2+2s}\right) \\ &= \epsilon_f m^{r(1/2-s)}(N/\gcd(N,m))^{1/2-s}\ell_0^{r(1/2-s)} \\ & \quad \times \sum_{(d,M)=1} \frac{G_{dlm,f}(1-s)}{G_{dlm,f}(s)} \frac{\psi(d)L_M(1-s,\tilde{f}\otimes\chi_d\chi_l\chi_m)\chi_{l'}(d)}{d^{w+r(s-1/2)}} \\ & \quad \times \prod_{p|M}\prod_{i=1}^{r}\left(1-\chi_d(p)\chi_l(p)\chi_m(p)\alpha_p^{(i)}p^{-s}\right)\left(1+\chi_d(p)\chi_m(p)\tilde{\alpha}_p^{(i)}p^{-1+s}\right), \end{aligned}$$

where ℓ_0 is either 1, 2, 4, or 8. Recall that the gamma factors depend only on f and the sign of dlm. We now use the χ_l' to recombine the right hand side as a sum of series with uniform signs, obtaining

$$\begin{aligned} &Z_M(s, w; \chi_{l'}, \chi_l\chi_m; f) \prod_{p|(M/m)} \prod_{i=1}^{r} \left(1 - (\tilde{\alpha}_p^{(i)})^2 p^{-2+2s}\right) \qquad (5.10) \\ &= m^{r(1/2-s)}(N/(N,m))^{1/2-s} \sum_{\substack{m'|(M/m) \\ (m',2)=1}} \sum_{l''} A^{(2)}_{m',l',l''}(s) \\ &\quad \times Z_M(1-s, w + r(s-1/2); \chi_{l'}\chi_{m'}, \chi_{l''}\chi_m; f), \end{aligned}$$

where the coefficients $A^{(2)}_{m',l',l''}(s)$ are analytic and satisfy the upper bound

$$A^{(2)}_{m',l',l''}(-\epsilon - it) \ll m'^{r\epsilon}(1+|t|)^{r(1+2\sigma+\epsilon)}.$$

Remark 5.11. *The trivial bound* $|\alpha_p^{(i)}| < p^{1/2}$ *on the size of the Fourier coefficients is used here to show that their size does not influence the bound for the* $A^{(2)}_{m',l',l''}$.

We now apply the w functional equation to the series $Z_M(s, w; \chi_{l'}\chi_{m'}, \chi_l\chi_m; f)$, obtaining

$$\begin{aligned} &Z_M(s, w; \chi_{l'}\chi_{m'}, \chi_l\chi_m; f) \prod_{p|(M/m')} \left(1 - p^{-2+2w}\right) \qquad (5.12) \\ &= M^{1/2-s} \sum_{\substack{m''|(M/m') \\ (m'',2)=1}} \sum_{l''} A^{(3)}_{m'',l',l''}(w) Z_M(w+s-1/2, 1-w; \chi_{l'}\chi_{m'}\chi_{m''}, \chi_{l''}\chi_m; f), \end{aligned}$$

where the coefficients $A^{(3)}_{m'',l',l''}(w)$ are analytic and satisfy the upper bound

$$A^{(3)}_{m'',l',l''}(-\sigma - it) \ll m''^{\epsilon}(1+|t|)^{1+2\sigma+\epsilon}.$$

The proposition follows after applying (5.9), (5.10), (5.12) in succession, as sketched in Section 2. Under these transformations of (s, w) we have, in the case $r = 2$ (for example),

$$(s, w) \to (s+w-1/2, 1-w) \to (3/2-s-w, w+2s-1) \to (s, 2-2s-w).$$

Thus, if we start with $s = 1/2$, $w = -\epsilon - it$ then (5.10) is applied to $(-\epsilon - it, 1 + \epsilon + it)$, and (5.12) is applied to $(1 + \epsilon + it, -\epsilon - it)$. Factors of $M^{1/2}$ are introduced at the second and third transformations. Note that a possible factor of $m^{1+2\epsilon}$ that could have been introduced by the second transformation is offset by a factor of $m^{-1-\epsilon}$ from the first transformation. A similar argument, as described at the end of

Section 2, does the cases $r = 1$ and $r = 3$. Thus, when $r = 1$ we obtain

$$\begin{aligned}(s, w) &\to (s + w - 1/2, 1 - w) \to M^{1-s-w}(3/2 - s - w, s)\\ &\to M^{3/2-2s-w}(1 - w, 1 - s).\end{aligned}$$

and when $r = 3$ we obtain

$$\begin{aligned}(s, w) &\to (s + w - 1/2, 1 - w) \to M^{1-s-w}(3/2 - s - w, 3s + 2w - 2)\\ &\to M^{7/2-4s-3w}(2s + w - 1, 3 - 3s - 2w)\\ &\to M^{5-6s-4w}(2 - 2s - w, w + 3s - 3/2)\\ &\to M^{5-6s-4w}(s, 5/2 - 3s - w)\\ &\to M^{11/2-7s-4w}(1 - s, 1 - w).\end{aligned}$$

To complete the proof of the proposition we apply the functional equations above in sequence to $Z^{(l,l')}(1/2, w)$, keeping track of the upper bounds for the $A^{(1)}_{m,l,l''}$, $A^{(2)}_{m',l',l''}$ and $A^{(3)}_{m",l',l''}$. For example, the M bound is obtained as follows. When $r = 1$:

$$(1/2, w) \to M^{1/2-w}(1 - w, 1/2),$$

when $r = 2$:

$$(1/2, w) \to M^{1-2w}(1/2, 1 - w),$$

and when $r = 3$:

$$(1/2, w) \to M^{2-4w}(1/2, 1 - w).$$

and the power of $|t|$ is obtained similarly. □

Several questions were left unanswered in Proposition 5.7: Does $Z^{(l,l')}(1/2, w)$ really have a pole at $w = 1$ (i.e is the order of the pole at least 1)? Also, if $Z^{(l,l')}(1/2, w)$ does have a pole of order $p \geq 1$ at $w = 1$, then what is a lower bound for the leading coefficient in the Laurent expansion of $Z^{(l,l')}(1/2, w)$?

The key ingredient in this coefficient is $L_M(2s, f, \mathrm{sym}^2)$, the symmetric square L-series of f with the Euler factors corresponding to primes dividing M removed. More precisely, the key ingredient is the leading coefficient of $L_M(2s, f, \mathrm{sym}^2)$ in the expansion about $s = 1/2$. If $L_M(2s, f, \mathrm{sym}^2)$ is analytic at $s = 1/2$ this is simply the value $L_M(1, f, \mathrm{sym}^2)$. This is because, referring to (5.2), a simple computation shows that for $s \neq 1/2$

$$\mathrm{Res}_{w=1} Z^{(l,l')}(s, w) = L_M(2s, f, \mathrm{sym}^2).$$

In the case $r = 1$ the polar part of $Z_1(1/2, w)$ near $w = 1$ is given by the sum of the contributions of the two potential polar lines $w = 1$ and $w + s - 1/2 = 1$ passing through $(1/2, 1)$. If $L(s, f)$ is a zeta function or a quadratic L-series then both lines are polar, $L_M(2s, f, \mathrm{sym}^2) =$

$\zeta_M(2s)$, $\epsilon = 1$, and there will be a double pole at the point $(1/2, 1)$ with leading coefficient κ_f bounded below by a constant independent of M. If $L(s, f)$ is a Dirichlet L-series with a character χ_N of order different from 1 or 2 then only $w = 1$ is polar and $L_M(2s, f, \mathrm{sym}^2) = L_M(2s, \chi_N^2)$. Thus in this case there will be a pole of order 1. The residue will be $\kappa_f \gg 1/\log N$ *unless* χ_N is a fourth order character and $L_M(2s, \chi_N^2)$ has a Siegel zero, in which case the best we can say is $\kappa_f \gg (\log N)^{1-\epsilon}/N^{1/2}$.

The case $r = 2$ and f irreducible cuspidal has been completely worked out in [FH95]. In this case, as long as $\epsilon \neq -1$, there is a single pole at $w = 1$ with residue a simple multiple of $\kappa_f = L(1, f, \mathrm{sym}^2)$. By [HL94] we know that $\kappa_f \gg 1/\log N$. If instead f is a lift from GL(1), then $L(s, f)$ is a product of two $GL(1)$ L-series, i.e $L(s, f) = L(s, \chi_{N_1})L(s, \chi_{N_2})$. In this case,

$$L_M(2s, f, \mathrm{sym}^2) = L(2s, \chi_{N_1}^2)L(2s, \chi_{N_2}^2)L(2s, \chi_{N_2}\chi_{N_1}).$$

If $\chi_{N_1}^2, \chi_{N_2}^2, \chi_{N_2}\chi_{N_1}$ are all non-trivial then the analysis is the same as the cuspidal analysis. There will be a pole of order 1 as long as $\epsilon \neq -1$. The residue will be bounded below by $\kappa_f \gg 1/(\log N)^3$ unless one of $L(2s, \chi_{N_1}^2)L(2s, \chi_{N_2}^2)L(2s, \chi_{N_2}\chi_{N_1})$ has a Siegel zero. If this happens the lower bound will be $\kappa_f \gg 1/((\log N)^{1+\epsilon}N^{1/2})$. (Note that we are applying a Siegel-Tatuzawa type argument here. We can assume a Siegel zero belongs to the χ with largest conductor and that the other two do not have Siegel zeros.)

If at least one of $\chi_{N_1}^2, \chi_{N_2}^2, \chi_{N_2}\chi_{N_1}$ is trivial then the residue analysis is simplified, as explained in [BFH04]. The key point is that if at least one is trivial then there is no chance of a cancellation of the highest order term. If all three are trivial there is a pole of order 4, with leading coefficient $\kappa_f \gg 1$. If two are trivial then there is a pole of order 3, with leading coefficient $\kappa_f \gg 1/\log N$ unless there is a Siegel zero, in which case $\kappa_f \gg (\log N)^{1-\epsilon}/N^{1/2}$. If one is trivial then $\kappa_f \gg 1/(\log N)^2$ unless there is a Siegel zero, in which case $\kappa_f \gg 1/((\log N)^{\epsilon}N^{1/2})$.

The case $r = 3$ and f cuspidal and $L(s, f, \mathrm{sym}^2)$ has a pole at $s = 1$ (i.e. when f is self-dual) has been completely worked out in [BFH04]. There is a pole of order 2 and the lower bound is

$$\kappa_f \gg \mathrm{Res}_{s=1} L(s, f, \mathrm{sym}^2),$$

which, by a result of [Sen04], is bounded below by $\kappa_f \gg 1/\log N$. The case $r = 3$ and f cuspidal but not self-dual is less well understood. In this case, $Z^{(l,l')}(1/2, w)$ has a pole of order 1 at $w = 1$. The analysis of

[Bru06] shows that if $\epsilon \neq -1$, then for some exponent $A > 0$,

$$L(1, f \times f) \gg N^{-A}.$$

This function factors as $L(1, f \times f) = L(1, f, \mathrm{sym}^2)L(1, \tilde{f})$, and using the crude bound $L(1, \tilde{f}) \ll N^{\varepsilon}$ gives

$$\kappa_f = L(1, f, \mathrm{sym}^2) \gg N^{-A}.$$

Note that if f is not self-dual, but $L(s, f, \mathrm{sym}^2)$ is the L-function of an automorphic form on GL(6) (as expected from the Langlands conjectures), we can use [HR95], Corollary 3.2, to rule out the Siegel zero, giving $L(1, f \times f) \gg 1/\log N$.

If $L(s, f)$ factors as a product of a $GL(2)$ cuspidal L-series times a $GL(1)$ L-series, i.e $L(s, f) = L(s, f_0)L(s, \chi)$, then

$$L(s, f, \mathrm{sym}^2) = L(s, f_0, \mathrm{sym}^2)L(s, f_0 \times \chi)L(s, \chi^2).$$

There is a pole of order 2 if χ^2 is trivial and by the results of [HL94] (see the appendix by Goldfeld, Hoffstein and Lieman) and [HR95], $\kappa_f \gg 1/(\log N)^2$. If χ^2 is not trivial then there is a pole of order 1 and by the same analysis, $\kappa_f \gg 1/(\log N)^3$ unless there is a Siegel zero, in which case $\kappa_f \gg 1/(N^{1/2}(\log N)^{1-\epsilon})$.

If $L(s, f)$ factors as a product of three $GL(1)$ L-series then the pole can have order between 1 and 7 as the pole of $L(s, f, \mathrm{sym}^2)$ can have order varying between 0 and 6. This is because in this case

$$\begin{aligned} L(s, f, \mathrm{sym}^2) &= L(s, \chi_{N_1}^2)L(s, \chi_{N_2}^2)L(s, \chi_{N_3}^2)L(s, \chi_{N_1}\chi_{N_2}) \\ &\quad \times L(s, \chi_{N_1}\chi_{N_3})L(s, \chi_{N_2}\chi_{N_3}). \end{aligned}$$

The worst lower bound is thus $\kappa_f > 1/(\log N)^6$ if there is no Siegel zero, and $\kappa_f > 1/(N^{1/2}(\log N)^{4+\epsilon})$ if there is a Siegel zero.

We summarize the above in

Proposition 5.13. *If $\epsilon \neq -1$ the series $Z^{(1,1)}(1/2, w)$ has a pole of of order at least 1 and at most 7 at $w = 1$. Denote the leading coefficient in the Laurent expansion of $Z^{(1,1)}(1/2, w)$ by κ_f. Then for some exponent $A > 0$, $\kappa_f \gg N^{-A}$. In all cases but the case $r = 3$ and f cuspidal but not self-dual, the value $A = 1/2+\epsilon$ will suffice. If it is furthermore true that $L(2s, f, \mathrm{sym}^2)$ is not divisible by any $L(2s, \chi)$, with χ quadratic, then $\kappa_f \gg 1/(\log N)^6$.*

Remark 5.14. In fact the lower bound $\kappa_f \gg N^{-A}$ is all that we require for our non-vanishing application. The rest of the analysis would be necessary if one were to go further and compute explicit constants or derive first moment results. We include it here for potential future applications, and for the record.

6. Proof of Theorem 1.1

The results of the previous sections can be summarized informally as follows. Let f be an automorphic form on $\mathrm{GL}(r)$, $r = 1, 2, 3$ of level N and let $\tilde{f}$ be its contragredient. Then there is a functional equation relating the Dirichlet series whose coefficients are the central twisted L-values of f,

$$\sum_d \frac{L(1/2, f \otimes \chi_d)}{d^w}$$

to that of its contragredient (as usual, the correction polynomials are suppressed). Actually, the relation requires a linear combination of such terms on both sides, where the L-function in the numerator can have an extra twist on the inside by some Dirichlet character, say, ψ, of conductor m, and can be multiplied on the outside by some other character, say, η:

$$\sum_d \frac{L(1/2, f \otimes \chi_d \psi)\eta(d)}{(d\, m)^w}.$$

Notice that the term in the denominator is at least as large as the conductor of the twist in the L-function in the numerator.

The number of terms in the linear combination on either side is at most N^{ε}, and each coefficient contains some Gamma factors times arithmetic data, which is also essentially independent of N, that is, bounded by N^{ε}. The functional equation has a "level," which is $N^{2\theta_r}$, where $\theta_r = 1/2, 1$, or 2 on $\mathrm{GL}(r)$, $r = 1, 2, 3$, respectively.

More precisely, we state the following

Theorem 6.1. *There exist some $J, \tilde{J} \ll N^{\varepsilon}$ for which the following holds. Let $j = 1, \ldots, J$ and $\tilde{j} = 1, \ldots, \tilde{J}$. For each j and $\tilde{j}$, there are characters ψ_j, $\tilde{\psi}_{\tilde{j}}$, η_j, $\tilde{\eta}_{\tilde{j}}$, and coefficients $\alpha_j(w)$ and $\tilde{\alpha}_{\tilde{j}}(w)$ containing Gamma factors are some local data. The characters ψ_j and $\tilde{\psi}_{\tilde{j}}$ have conductors m_j and $\tilde{m}_{\tilde{j}}$, respectively. The coefficients $\alpha_j(w)$ and $\tilde{\alpha}_{\tilde{j}}(w)$ are $\ll N^{\varepsilon}$. Define the Dirichlet series*

$$Z_j(f, w) := \sum_d \frac{L(1/2, f \otimes \chi_d \psi_j)\eta_j(d)}{(d\, m_j)^w},$$

and similarly for $\tilde{Z}_{\tilde{j}}$. Then the following functional equation holds:

$$N^{\theta_r w} \sum_{j=1}^{J} \alpha_j(w) Z_j(f, w) = N^{\theta_r(1-w)} \sum_{\tilde{j}=1}^{\tilde{J}} \tilde{\alpha}_{\tilde{j}}(1-w) \tilde{Z}_{\tilde{j}}(\tilde{f}, 1-w), \quad (6.2)$$

where $\theta_1 = 1/2$, $\theta_2 = 1$, and $\theta_3 = 2$ corresponding to $\mathrm{GL}(r)$, $r = 1, 2, 3$.

Recall the calculation leading to (4.4), which demonstrates the simplest example of this identity.

Let $Z^*(f,w)$ denote the sum on the left hand side of (6.2), and let $\tilde{Z}^*(\tilde{f},1-w)$ be the sum on the right hand side, so that

$$N^{\theta_r w} Z^*(f,w) = N^{\theta_r(1-w)}\tilde{Z}^*(\tilde{f},1-w). \tag{6.3}$$

Integrating (6.3) and estimating the resulting inverse Mellin transform of products of Gamma factors, we obtain

$$\frac{1}{2\pi i}\int_{(2)} N^{(w+1/2)\theta_r} Z^*(f,w+1/2)dw = \sum_j \sum_d \frac{L(1/2, f\otimes \chi_d \psi_j)\eta_j(d)}{(d\,m_j)^{1/2}} V_j\left(\frac{d\,m_j}{N^{\theta_r}}\right),$$

where $V_j(x)$ decays exponentially for $x \gg_j 1$.

Pull the contour back to $\mathfrak{Re}(w) = -1$, passing through the poles of Z^*.

Let $\mathcal{R}$ denote the residual contribution, use the functional equation (6.2), and make the change of variables $w \mapsto -w$:

$$\begin{aligned}\mathcal{R} \;=\; & \sum_j \sum_d \frac{L(1/2, f\otimes \chi_d \psi_j)\eta_j(d)}{(d\,m_j)^{1/2}} V_j\left(\frac{d\,m_j}{N^{\theta_r}}\right) \\ & + \sum_{\tilde{j}} \sum_d \frac{L(1/2, \tilde{f}\otimes \chi_d \tilde{\psi}_{\tilde{j}})\tilde{\eta}_{\tilde{j}}(d)}{(d\,\tilde{m}_{\tilde{j}})^{1/2}} \tilde{V}_{\tilde{j}}\left(\frac{d\,\tilde{m}_{\tilde{j}}}{N^{\theta_r}}\right).\end{aligned}$$

By Proposition 5.7, $\mathcal{R} \gg N^{-A}$ for some $A > 0$. Assume by contradiction that all of the twisted L-functions vanish at the center for all twists $d\,m_j$ and $d\,\tilde{m}_{\tilde{j}} \ll N^{\theta_r+\varepsilon}$. Then as the cutoff functions V_j and $\tilde{V}_j$ decay exponentially, the right hand side above is exponentially small in N, while the left hand side is bounded from below by some polynomial. This contradiction completes the proof of Theorem 1.1.

7. The "Moment Approach"

As the details of this approach appear in [Iwa90], we only give a sketch of the results claimed in §1.3, in the case rank $r = 2$ in the weight k aspect; the others are similar.

Let f be an automorphic form on GL(2) of weight k and consider the sum

$$M(X) := \sum_d L(1/2, f, \chi_d)\Psi\left(\frac{d}{X}\right), \tag{7.1}$$

where Ψ is a smooth non-negative function with support contained in the interval $[1, 2]$, say.

The main result of this section is

Theorem 7.2.

$$\begin{aligned} M(X) \ &\approx\ L(1, \mathrm{sym}^2 f)\cdot X \\ &\quad +O\left(X^{3/4}k^{-1/4}L(1/2, \mathrm{sym}^2 f) + X^{1/2}k^{1/2} + k^{3/2}\right). \end{aligned}$$

It is easy to conclude from this theorem the promised non-vanishing result. Namely, using the Siegel zero analysis $L(1, \mathrm{sym}^2 f) \gg 1/\log k$ and the convexity bound $L(1/2, \mathrm{sym}^2 f) \ll k^{1/2+\varepsilon}$, the main term dominates the error as soon as $X \gg k^{3/2+\varepsilon}$. But this implies that all central L-values in (7.1) cannot vanish for all $d \ll X$. Hence there is a nonvanishing twist with $|d| \ll k^{3/2+\varepsilon}$. This matches the corresponding entry in the table of §1.3.

The rest of this section is devoted to proving Theorem 7.2. The approximate functional equation is

$$L\left(\frac{1}{2}, f, \chi_d\right) \ \approx\ \sum_m \frac{\lambda(m)\chi_d(m)}{\sqrt{m}} V\left(\frac{m}{k\,d}\right), \tag{7.3}$$

plus a similar term with the root number and the dual form. Here V is a smooth cutoff function, essentially supported in $(0, 1)$. If we were instead interested in the level aspect, then we would need to treat the second term separately, as the root number adds extra complications in the N dependence. As we are currently seeking the k dependence, we will ignore the term with the root number.

Inserting (7.3) into (7.1) gives:

$$M(X) = \sum_m \frac{\lambda(m)}{\sqrt{m}} \sum_d \chi_d(m)\Psi\left(\frac{d}{X}\right) V\left(\frac{m}{k\,d}\right). \tag{7.4}$$

Break (7.4) into $\Sigma_1 + \Sigma_2$, where the first sum is over square values of m.

7.1. **Estimating Σ_1.** There is no cancellation here from the character since $\chi_d(m^2)=\chi_d(m)^2=1$, so

$$\Sigma_1 \approx \sum_{X<d<2X}\sum_{m<(k\,X)^{1/2}} \frac{\lambda(m^2)}{m} = X \sum_{m<(k\,X)^{1/2}} \frac{\lambda(m^2)}{m}. \tag{7.5}$$

Let

$$Y=(k\,X)^{1/2}. \tag{7.6}$$

Consider

$$\sum_{m<Y}\frac{\lambda(m^2)}{m} = \frac{1}{2\pi i}\int_{(2)} L(1+s,\mathrm{sym}^2 f)Y^s ds/s.$$

Pull the contour to $\mathfrak{Re}(s)=-1/2$, picking up a pole at $s=0$:

$$\begin{aligned}\sum_{m<Y}\frac{\lambda(m^2)}{m} &= L(1,\mathrm{sym}^2 f)+\frac{1}{2\pi i}\int_{(-1/2)} L(1+s,\mathrm{sym}^2 f)Y^s ds/s\\ &= L(1,\mathrm{sym}^2 f)+O\left(L(\frac{1}{2},\mathrm{sym}^2 f)Y^{-1/2}\right). \qquad (7.7)\end{aligned}$$

7.2. **Bounding Σ_2.** Now we apply quadratic reciprocity, Gauss sums and Poisson summation.

Return to (7.4). For the terms in Σ_2, namely $m\neq\square$, use quadratic reciprocity to turn $\chi_d(m)$ into $\chi_m(d)$, then use Gauss sums to lift d to a real variable:

$$\chi_m(d)=\frac{1}{\tau(\bar\chi_m)}\sum_{a(\mathrm{mod}\,m)}\bar\chi_m(a)e^{2\pi i a d/q}.$$

Then breaking the m sum into dyadic intervals, using $|\tau(\chi_m)|=m^{\frac{1}{2}}$, and applying Poisson summation gives

$$\begin{aligned}\Sigma_2 &\ll \sum_{\substack{m\neq\square\\ m\sim k\,X}}\frac{\lambda(m)}{\sqrt{m}}\frac{1}{\tau(\bar\chi_m)}\sum_{a(\mathrm{mod}\,m)}\bar\chi_m(a)\sum_{d\in\mathbb{Z}}e^{2\pi i a d/m}\Psi\left(\frac{d}{X}\right) \qquad (7.8)\\ &\ll \sum_{\substack{m\neq\square\\ m\sim k\,X}}\frac{\lambda(m)}{m}\sum_{a(\mathrm{mod}\,m)}\bar\chi_m(a)\sum_{\ell\in\mathbb{Z}}\int_{\mathbb{R}}e^{2\pi i a x/m}\Psi\left(\frac{x}{X}\right)e^{-2\pi i x\ell}dx\\ &\ll (k\,X)^{-1}\sum_{\substack{m\neq\square\\ m\sim k\,X}}\lambda(m)\sum_{a(\mathrm{mod}\,m)}\bar\chi_m(a)\sum_{\ell\in\mathbb{Z}}X\int_{\mathbb{R}}\Psi(x)\,e^{2\pi i x X(a/m-\ell)}dx\\ &= k^{-1}\sum_{\substack{m\neq\square\\ m\sim k\,X}}\lambda(m)\sum_{a(\mathrm{mod}\,m)}\bar\chi_m(a)\sum_{\ell\in\mathbb{Z}}\widehat{\Psi}\left(X(a/m-\ell)\right),\end{aligned}$$

by partial summation.

Assuming Ψ is smooth, the transform $\widehat{\Psi}$ has rapid decay, and thus the sum over ℓ is negligible except for $\ell = 0$. Similarly, the terms with $a > m/X$ do not contribute. Thus

$$\begin{aligned}\Sigma_2 &\ll k^{-1} \sum_{\substack{m \neq \square \\ m \sim k\,X}} \lambda(m) \sum_{a < m/X} \chi_m(a) \\ &\ll k^{-1} \sum_{a \sim k} \sum_{\substack{m \neq \square \\ m \sim k\,X}} \lambda(m)\chi_a(m),\end{aligned}$$

using quadratic reciprocity yet again and breaking the a sum into dyadic intervals.

Next apply the functional equation to the inner m sum. The conductor of the sum is k^2a^2 and its length is $k\,X$. Thus

$$\sum_{m \sim k\,X} \lambda(m)\chi_a(m) \approx \frac{k\,X}{k\,a} \sum_{m \sim \frac{k^2a^2}{k\,X}} \lambda(m)\chi_a(m), \tag{7.9}$$

which we insert into (7.10):

$$\Sigma_2 \ll X\,k^{-2} \sum_{a \sim k} \sum_{m \sim k^3 X^{-1}} \lambda(m)\chi_a(m),$$

by a further dyadic decomposition.

Apply Cauchy-Schwarz in the a variable:

$$\Sigma_2 \ll X k^{-3/2} \left(\sum_{a \sim k} \left| \sum_{\substack{m \neq \square \\ m \sim k^3\,X^{-1}}} \lambda(m)\chi_a(m) \right|^2 \right)^{1/2}. \tag{7.10}$$

Recall Heath-Brown's real character sum estimate [HB95]:

$$\sum_{\substack{a \neq \square \\ a \sim A}} \left| \sum_{\substack{m \neq \square \\ m \sim B}} \lambda(m) \left(\frac{a}{m}\right) \right|^2 \ll (A+B) \sum_{\substack{m \neq \square \\ m \sim B}} |\lambda(m)|^2 \ll B(A+B).$$

Applied to (7.10) (one can arrange for the a sum to go over square-free numbers) this gives

$$\Sigma_2 \ll k^{1/2} X^{1/2} + k^{3/2} \tag{7.11}$$

Combining (7.5), (7.6), (7.7) with (7.11) gives the estimate claimed in §1.3.

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